AN9705
Rev.0.00
Jun 1, 1997

A Theoretical View of Coherent Sampling

Introduction

In recent years, the comprehensive science behind testing the performance of A/D converters has been perfected. Commercially available equipment used to test the performance of A/D converters has been forced to keep up with the various performance measures developed. Some of these parameters include effective number of bits (ENOB), total harmonic distortion (THD), and signal to noise ratio (SNR). A number of data acquisition systems (DAS) have been to developed to test the performance of these A/D converters.

One approach for measuring the parameters listed above is to use frequency-based continuous wave tests on A/D converters. Since these tests perform fast Fourier (FFT) transforms [1] of the sample signal, one issue arises. The continuous (sinusoidal) wave must be sampled coherently by the DAS system in order to avoid FFT artifacts. This application note was written to assist those trying to understand coherent sampling mathematically. For a more general discussion on coherent sampling, refer to [2].

Definition of Coherent Sampling

In order to avoid FFT artifacts, the ratio between the frequency of the input signal and the sampling frequency of the system must be able to be expressed as a rational number. Let us take the ideal sinusoid formed from N samples:

$$x(n) = \sin(2\pi f\, n) \qquad \text{(EQ. 1)}$$

where n is defined as the sampling index $0 \leq n \leq N - 1$.

f is equal to the digital frequency. The analog frequency, F, is defined as $F = F_S\, f$, where F_S is the sampling frequency of the system. The digital frequency is considered coherent if f is a rational number; since there are only N samples in the sinusoidal data set, f is the rational number

$$f = \frac{k_0}{N} = \frac{F}{F_S}, \qquad \text{(EQ. 2)}$$

where k_0 and N are integers.

$$x(n) = \sin\left(\frac{2\pi}{N}k_0 n\right) \qquad \text{(EQ. 3)}$$

where n is defined as the sampling index $0 \leq n \leq N - 1$.

Mathematically, there is no reason why f is required to be rational; N remains an integer, but k_0 can range from all real numbers. In the following section, we will derive the discrete Fourier transform (DFT) of Equation 3 without requiring Equation 2 to be rational.

Mathematical Reason Behind Coherent Sampling

The DFT is defined as

$$X(k) = \sum_{n=0}^{N-1} x(n) e^{-j2\pi k\frac{n}{N}} \qquad \text{(EQ. 4)}$$

where $0 \leq k \leq N\text{-}1$ and k is the frequency sampling index. It is related to the analog frequency by

$$F = \frac{k}{N} F_S \qquad \text{(EQ. 5)}$$

To clear up a point of confusion, the FFT is an efficient method of computing DFT if N, the number of samples, is a power of 2. The DFT and FFT are only a tools used to identify the spectral purity of a periodic tone. Sinusoids certainly have the simplest spectral results (with which we will derive later); but other periodic signals, such as triangle waves and square waves, also have distinct spectral response. These transforms were invented by mathematicians at the turn of the century as a simple extension of the unit circle/sine-cosine relationship. As engineers, we apply the DFT and FFTs to a wide variety of signals and have been trained to identify a signal, perform signal-to-noise ratios, and evaluate the quality of whole systems.

Notice that F is now a discretely sampled domain related to the sampling frequency. k, also known as the frequency bin can only go up to N-1, and thus the maximum frequency is $(N\text{-}1)F_S/N$.

Recall that one of the Euler's identities is

$$\sin(x) = \frac{e^{jx} - e^{-jx}}{2j} \qquad \text{(EQ. 6)}$$

Using Equations 3, 4, and 6, the following can be derived:

$$X(k) = \frac{1}{2j}\sum_{n=0}^{N-1}\left(e^{j\frac{2\pi}{N}k_0 n} - e^{-j\frac{2\pi}{N}k_0 n}\right)e^{-j2\pi k\frac{n}{N}} \qquad \text{(EQ. 7)}$$

$$X(k) = \frac{1}{2j}\sum_{n=0}^{N-1}\left(e^{j\frac{2\pi}{N}(k_0 - k)n} - e^{-j\frac{2\pi}{N}(k_0 + k)n}\right). \qquad \text{(EQ. 8)}$$

The summation can be easily evaluated using the geometric series equation

$$\sum_{n=0}^{N-1} a^n = \frac{1 - a^N}{1 - a} \text{ for } |a| < 1 \qquad \text{(EQ. 9)}$$

Thus Equation 8 becomes

$$X(k) = \frac{1}{2j}\left(\frac{1-e^{j\frac{2\pi}{N}(k_0-k)N}}{1-e^{j\frac{2\pi}{N}(k_0-k)}} - \frac{1-e^{-j\frac{2\pi}{N}(k_0+k)N}}{1-e^{-j\frac{2\pi}{N}(k_0+k)}}\right) \qquad \text{(EQ. 10)}$$

Take advantage of the fact that

$$e^{2N} = e^N e^N \qquad \text{(EQ. 11)}$$

Equation 10 can be written as

$$X(k) = \frac{1}{2j}\left(\left[\frac{e^{-j\pi(k_0-k)} - e^{j\pi(k_0-k)}}{e^{-j\frac{\pi}{N}(k_0-k)} - e^{j\frac{\pi}{N}(k_0-k)}}\right]\frac{e^{j\pi(k_0-k)}}{e^{j\frac{\pi}{N}(k_0-k)}} - \left[\frac{e^{j\pi(k_0+k)} - e^{-j\pi(k_0+k)}}{e^{j\frac{\pi}{N}(k_0+k)} - e^{-j\frac{\pi}{N}(k_0+k)}}\right]\frac{e^{-j\pi(k_0+k)}}{e^{-j\frac{\pi}{N}(k_0+k)}}\right) \qquad \text{(EQ. 12)}$$

The resulting bracket terms can be reformed into sine wave using the synthesis of Equation 6. Thus Equation 12 can be written as

$$X(k) = \frac{1}{2j}\left(\left[\frac{\sin[\pi(k_0-k)]}{\sin\left[\frac{\pi}{N}(k_0-k)\right]}\right]e^{j\pi\left[\frac{N-1}{N}\right](k_0-k)} - \left[\frac{\sin[\pi(k_0+k)]}{\sin\left[\frac{\pi}{N}(k_0+k)\right]}\right]e^{-j\pi\left[\frac{N-1}{N}\right](k_0+k)}\right) \qquad \text{(EQ. 13)}$$

for k = 0,1,...,N-1.

Now the importance of the digital frequency, f, being rational becomes clear. If f is rational, then k_0 is an integer, and the numerators of both sin(x)/sin(x/N) functions are 0 for all frequency index k. Remember that the frequency index, k, is also related to a discrete analog frequency bin as shown in Equation 5. Now note that the denominator of the first sin(x)/sin(x/N) function goes to sin(0) = 0 only when $k = k_0$. Thus the signal is indeterminate when $k = k_0$. The first sin(x)/sin(x/N) function at this indeterminate point is equal to N using L'Hospital rule. A similar indeterminate function occurs for the second sin(x)/sin(x/N) function when $k = N - k_0$. The denominator is equal to $\sin(\pi) = 0$. Thus, if k_0 is an integer, then Equation 13 is

$$X(k) = \frac{N}{2j}\left(\delta(k-k_0) + \delta(k-N+k_0)\right) \text{ for } k = 0,1,\ldots,N\text{-}1, \qquad \text{(EQ. 14)}$$

where $\delta(x)$ is a delta function defined by $\delta(x) = 0$ for all x except for x = 0; $\delta(0) = 1$. Figure 1 shows a plot of Equation 13; the second delta function is not identical to Equation 14 due to the sinusoidal roll-over limitations of the software. The pseudo-code used to generate this plot is given in the Appendix.

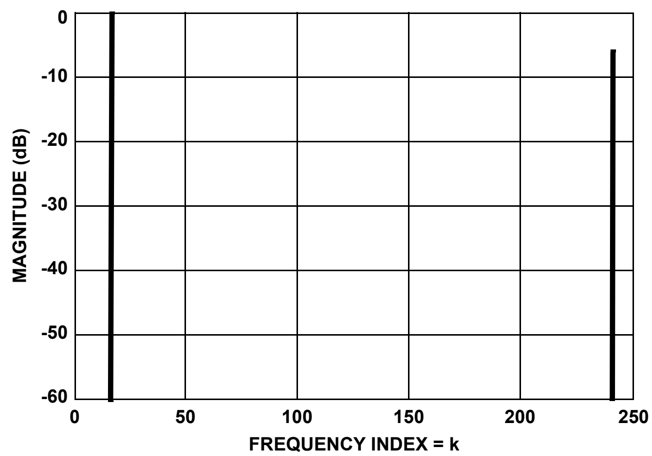

FIGURE 1. PLOT OF THE FREQUENCY RESPONSE OF A SINUSOID WHERE k_0 = 16, N = 256

We define coherent frequencies as when the digital frequency of the sinusoid is rational and the FFT results in perfect delta functions occurring at the frequency bin k_0 and $N - k_0$. Note that for logarithmic magnitude display of Equation 13, engineers often normalize the FFT and DFT results by 1/N. The number of samples, N, is seen as a computational gain. Note that the 2j term in the gain is a phase component. The magnitude plots of Equation 13 show that the energy of the sinusoid is divided between the two delta functions.

Now let us examine sinusoids where k_0 is not equal to an integer, and the function is not coherent with respect to N and F_S. The numerator of both sin(x)/sin(x/N) functions in Equation 13 can never go to 0 because k- k_0 can never equal 0 and k-N+ k_0 can never equal N. In fact, the linear domain has, for large N, a sinc function shifted about k_0 and N- k_0 frequency bins. The dB plots of Equation 13 with a non-integer k_0 reveal spreading about these same frequency bins. Figure 2 shows a 256 point sinusoid with a k_0 = 15.25. The pseudo-code used to generate this plot is given in the Appendix.

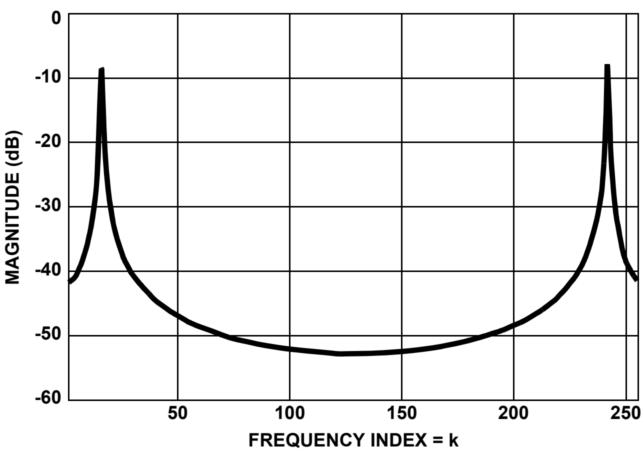

FIGURE 2. PLOT OF THE FREQUENCY RESPONSE OF A SINUSOID WHERE k_0 = 15.25, N = 256

Summary

For a data set of N, it was shown that the ratio of F/F_S must have an equivalent ratio k_0/N that is a rational number. If this condition is not met, smearing across the frequency bins occurs. The DAS system is left with three options. First, it can compensate for the frequency artifact caused by non-coherent sampling using windowing. The compensation of non-coherent sampling can only be marginal, however, if the DAS system is limited in registers and computational capability. The second option is for the DAS system to fix the sampling frequency of the system, compute a frequency of the continuous wave that results in an equivalent ratio F/F_S = k_0/N that is rational, and tune the input continuous wave to the computed frequency. The third option is for the DAS system to fix the continuous wave frequency, compute a sampling frequency of the system that results in an equivalent ratio F/F_S = k_0/N that is rational, and tune the sampling frequency to the computed frequency. The latter two options are practical approaches for most DAS systems.

References

For Intersil documents available on the web, see http://www.semi.intersil.com/
Intersil AnswerFAX (321) 724-7800.

[1] JG Proakis, DG Manolakis, *Digital Signal Processing Principles, Algorithms, and Applications*, Prentice Hall, NJ., 1996.

[2] A. Aude, *AN9675, Coherent and Windowed Sampling with A/D Converters*, Intersil, AnswerFAX doc. #99675.

Appendix

Pseudo-code used to generate Figure 1 from Equation 13

```
k = 0:255;
k0 = 16;
A1 = sin(pi*(k0-k)); A2 = sin(pi*(k0+k));
B1 = sin(pi*(k0-k)/256); B2 = sin(pi*(k0+k)/256);
C1 = exp(j*pi*255/256*(k0-k));
C2 = exp(-j*pi*255/256*(k0+k));
for o = 1:256
        if B1(o) ~= 0
            X1(o) = 1/(2*j)*(A1(o)/B1(o)*C1(o) -
            A2(o)/B2(o)*C2(o));
        else
            X1(o) = 256;
        end
end
plot(20*log10(abs(X1/256)),'w')
axis([1 256 -60 0])
xlabel('frequency index = k')
ylabel('Magnitude (dB)')
```

Pseudo-code used to generate Figure 2 from Equation 13

```
k=0:255;
k0 = 15.25;
A1 = sin(pi*(k0-k)); A2 = sin(pi*(k0+k));
B1 = sin(pi*(k0-k)/256); B2 = sin(pi*(k0+k)/256);
C1 = exp(j*pi*255/256*(k0-k));
C2 = exp(-j*pi*255/256*(k0+k));
X1 = 1/(2*j)*(A1./B1.*C1 - A2./B2.*C2);
plot(k, 20*log10(abs(X1/256)),'w')
axis([0 255 -60 0])
xlabel('frequency index = k')
ylabel('Magnitude (dB)')
```

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Renesas Electronics America Inc.
1001 Murphy Ranch Road, Milpitas, CA 95035, U.S.A.
Tel: +1-408-432-8888, Fax: +1-408-434-5351

Renesas Electronics Canada Limited
9251 Yonge Street, Suite 8309 Richmond Hill, Ontario Canada L4C 9T3
Tel: +1-905-237-2004

Renesas Electronics Europe Limited
Dukes Meadow, Millboard Road, Bourne End, Buckinghamshire, SL8 5FH, U.K
Tel: +44-1628-651-700, Fax: +44-1628-651-804

Renesas Electronics Europe GmbH
Arcadiastrasse 10, 40472 Düsseldorf, Germany
Tel: +49-211-6503-0, Fax: +49-211-6503-1327

Renesas Electronics (China) Co., Ltd.
Room 1709 Quantum Plaza, No.27 ZhichunLu, Haidian District, Beijing, 100191 P. R. China
Tel: +86-10-8235-1155, Fax: +86-10-8235-7679

Renesas Electronics (Shanghai) Co., Ltd.
Unit 301, Tower A, Central Towers, 555 Langao Road, Putuo District, Shanghai, 200333 P. R. China
Tel: +86-21-2226-0888, Fax: +86-21-2226-0999

Renesas Electronics Hong Kong Limited
Unit 1601-1611, 16/F., Tower 2, Grand Century Place, 193 Prince Edward Road West, Mongkok, Kowloon, Hong Kong
Tel: +852-2265-6688, Fax: +852 2886-9022

Renesas Electronics Taiwan Co., Ltd.
13F, No. 363, Fu Shing North Road, Taipei 10543, Taiwan
Tel: +886-2-8175-9600, Fax: +886 2-8175-9670

Renesas Electronics Singapore Pte. Ltd.
80 Bendemeer Road, Unit #06-02 Hyflux Innovation Centre, Singapore 339949
Tel: +65-6213-0200, Fax: +65-6213-0300

Renesas Electronics Malaysia Sdn.Bhd.
Unit 1207, Block B, Menara Amcorp, Amcorp Trade Centre, No. 18, Jln Persiaran Barat, 46050 Petaling Jaya, Selangor Darul Ehsan, Malaysia
Tel: +60-3-7955-9390, Fax: +60-3-7955-9510

Renesas Electronics India Pvt. Ltd.
No.777C, 100 Feet Road, HAL 2nd Stage, Indiranagar, Bangalore 560 038, India
Tel: +91-80-67208700, Fax: +91-80-67208777

Renesas Electronics Korea Co., Ltd.
17F, KAMCO Yangjae Tower, 262, Gangnam-daero, Gangnam-gu, Seoul, 06265 Korea
Tel: +82-2-558-3737, Fax: +82-2-558-5338

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