Introduction

This application note derives two fundamental curves for parallel resonant crystals which have direct bearing on designing for frequency accuracy and stability of parallel resonant mode based crystal oscillators and VCXOs. First is the load curve, the crystal parallel resonant frequency as a function of external circuit capacity. This is a non-linear curve of the frequency offset of a crystal as a function of load capacity. Second is the tuning curve, which is the inverse of the load curve. The tuning curve gives the load capacity as a function of the crystal frequency and is used to center an oscillator's frequency.
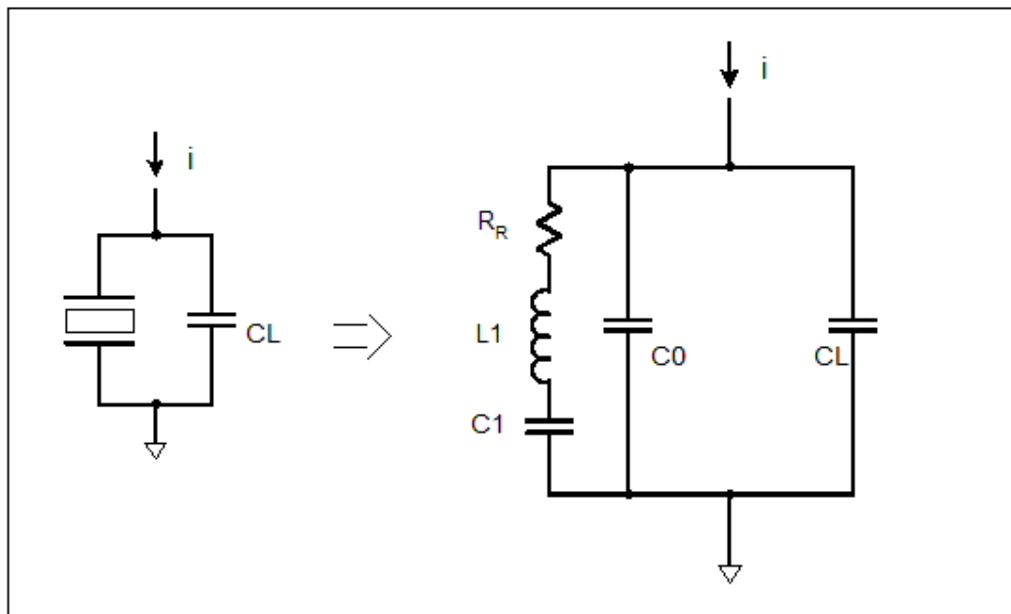
Typically these non-linear functions are approximated by their slopes at the nominal crystal frequency. This approximation forces the oscillator designer into a series of tuning capacitor trials to determine the set of nominal capacitors to center the frequency of their oscillator design. Possession of the exact tuning curve in conjunction with measured crystal data allows for direct solution of tuning capacitors, with the bias of the crystal tolerance removed, with an accuracy limited only by the measured data and the fit of the crystal electrical elements to the data. Further the load curve is required in order to predict the tuning range of a given VCXO oscillator for any particular crystal.

Modeling a Crystal in Parallel Resonant Mode

Modeling the frequency dependence of a crystal oscillator on load capacity requires the impedance of the crystal in the oscillator feedback loop. The impedance is simplified for most purposes by the network of Figure 1. The motional electrical elements are commonly named L_i, C_i and R_i where the i script denotes the vibrational mode of the crystal; 1 for fundamental, 3 for third overtone etc. There is one element, C_0, which is the static crystal holder capacity and is common to all modes.

The nomenclature used by crystal vendors for fundamental mode crystals, L_1, C_1, R_R (rather than R_1) and C_0, is adopted for this article for direct correspondence with measured crystal data. Additionally crystal vendors can provide FR for the low impedance motional series crystal resonant frequency and FL as the high impedance parallel resonant frequency for a series of external load capacitors, CL. Both frequencies are provided as either mode may be used depending on the application.

Figure 1. Electrical Model of a Crystal with a Parallel Load Capacitor

The following will help to familiarize the user with the physical basis of the electrical elements of the model.

The electrical elements of the impedance are directly related to the mechanical properties of the crystal when vibrating in a particular mode; mass, stiffness and damping coefficient. There are two reasons why this electrical to mechanical association is possible. The first is the well known correspondence of F=ma for mechanical systems and e = Ldi/dt for electrical systems. This formal correspondence immediately leads to associating force and voltage, position displacement and electrical charge, mass with inductance, damping with resistance and spring constant with capacity. The second is what makes the crystal special; when a crystal is physically vibrating the piezoelectric effect exhibited by quartz results in polarization charges appearing at the crystal electrodes in direct proportion to the crystal positional displacement.

The mechanical to electrical analogy for a crystal is then not just a formality with no practical application; rather the crystal behaves simultaneously as a mechanical and electrical element. Not only does the piezoelectric effect allow for electrical systems to take advantage of the much higher Q values available from mechanical systems, the inverse piezoelectric effect allows for stresses and therefore vibrations to be induced into crystals electrically. The inverse effect is used in actuators such as inkjet print injectors and slip-stick stepper motors where two orthogonal vibration modes are simultaneously driven electrically.

Crystal Series and Parallel Resonant Mode Frequencies

The following definitions tare used to simplify the subsequent algebra:

$$C_p = C0 + CL$$

$$Z_m = \frac{1}{sC1} + RR + sL1$$

$$C_T = C0 + C1 + CL$$

$$= C0 + C_p$$

$$= \frac{1 + sRC1 + s^2 L1C1}{sC1}$$

$$C_S = \frac{C0 C_p}{C0 + C_p}$$

$$= \frac{N(s)}{sC1}$$

$$Y_p = sC_p$$

C_S is the series combination of the crystal motional capacitance C1 and the total parallel capacity seen by the crystal, C0+CL. This is not the effective capacity of the series resonant mode, which is C1; it is the effective capacity of the series combination of C0 and C_P that with L1, determines the frequency of the parallel resonant mode.

The crystal circuit model is written as the parallel combination of the crystal series impedance and the parallel capacities.

$$Z_{XTAL} = \frac{Z_p Z_m}{Z_p + Z_m}$$

$$= \frac{Z_m}{1 + Y_p Z_m}$$

$$Z_{XTAL} = \frac{N(s)/sC1}{1 + sC_p * N(s)/sC1}$$

$$= \frac{N(s)}{sC1 + sC_p\left(1 + sRC1 + s^2 L1C1\right)}$$

$$= \frac{N(s)}{sC1 + sC_p + s^2 RC1C_p + s^3 L1C1C_p}$$

$$= \frac{1}{s(C1 + C_p)} \frac{1 + sRRC1 + s^2 L1C1}{\left(1 + s\dfrac{RRC1C_p}{(C1 + C_p)} + s^2 \dfrac{L1C1C_p}{(C1 + C_p)}\right)}$$

$$Z_{XTAL} = \frac{1}{sC_T} \frac{1 + sRRC1 + s^2 L1C1}{\left(1 + sRRC_s + s^2 L1C_s\right)} \qquad \textbf{Equation 1}$$

The crystal impedance is given by Equation 1 above. The series motional resonant mode frequency is defined by the zeros and is a property of the crystal only. The series resonant frequency is defined as FR and given by Equation 2 below. The parallel resonance frequency, FL, depends on the crystal, crystal package and the external load capacity is related to FR by **Equation 3** below. The simple impedance pole at the origin is simply the sum of all the crystal circuit capacities and like RR, is not relevant to tuning. The two resonant frequencies are related and as shown below; the parallel resonant frequency is higher than the series resonant frequency.

$$F_R = \frac{1}{2\pi} \frac{1}{\sqrt{L1C1}} \qquad \textbf{Equation 2}$$

$$F_L = \frac{1}{2\pi} \frac{1}{\sqrt{L1C_s}} = F_R \sqrt{1 + \frac{C1}{(C0 + CL)}} \qquad \textbf{Equation 3}$$

Derivation of the Sensitivity Based Load and Tuning Curves

Before deriving the exact crystal load and tuning curves below, it is useful to start with the tuning curve typically used, the tune sensitivity, otherwise known as the trim sensitivity. This is the derivative of the parallel resonant frequency with respect to CL when the crystal is loaded with the vendor specified nominal CL, CLn. It is easiest to begin by squaring Equation 3.

$$F_L^2 = F_S^2 \left(\frac{(C0 + CL)}{(C0 + C1 + CL)}\right)^{-1}$$

$$2F_L dF_L = F_S^2 \left(\frac{1}{C0 + CLn} - \frac{C0 + C1 + CLn}{(C0 + CLn)^2}\right) dCL$$

$$\frac{dF_L}{f_L} = -\frac{1}{2} \frac{C1(C0 + C1 + CLn)}{(C0 + CLn)^3} dCL$$

The sensitivity based load curve, Equation 4, follows directly by substituting finite differences for the differentials and making the further approximation to the first derivative approximation that C1 is small relative to C0+CL.

$$\frac{\Delta F_L}{F_L} = -\frac{1}{2}\frac{C1}{(C0+CLn)^2}\Delta CL \qquad \textbf{Equation 4}$$

Generating the tuning curve, Equation 5, the change in CL necessary to tune the crystal oscillator back to the nominal frequency, follows directly by substituting - ΔCL for ΔCL and rearranging.

$$\Delta CL = 2\frac{(C0+CLn)^2}{C1}\frac{\Delta F_L}{F_L} \qquad \textbf{Equation 5}$$

Derivation of the Crystal Load Curve

To generate the exact tuning curves, begin by expanding the frequency of the parallel resonance and associated CL in finite differences. The load resonant frequency then is FL = FLn+ ΔFL and the load capacity, CL, expanded from the nominal load capacity, CLn, is CLn+ ΔCL.

$$\frac{F_L+\Delta F_L}{F_L} = \frac{\dfrac{1}{\sqrt{\dfrac{(C0+CLn+\Delta CL)}{(C0+C1+CLn+\Delta CL)}}}}{\dfrac{1}{\sqrt{\dfrac{(C0+CLn)}{(C0+C1+CLn)}}}}$$

$$1+\frac{\Delta F_L}{F_L} = \sqrt{\frac{(C0+CLn)}{(C0+C1+CLn)}\frac{(C0+C1+CLn+\Delta CL)}{(C0+CL+\Delta CLn)}}$$

Next, square to make the subsequent manipulations of the RHS more apparent.

$$\left(1+\frac{\Delta F_L}{F_L}\right)^2 = \frac{(C0+CLn)}{(C0+CLn+\Delta CL)}\frac{(C0+C1+CLn+\Delta CL)}{(C0+C1+CLn)}$$

$$= \left(\frac{1}{1+\dfrac{\Delta CL}{(C0+CLn)}}\right)\left(1+\frac{\Delta CL}{(C0+CLn)}\left(\frac{C0+CLn}{C0+C1+CLn}\right)\right)$$

Make substitutions using ε and ε' to expand the RHS in terms of ΔCL. In general, ΔCL is not small relative to CLn.

$$\varepsilon = -\frac{\Delta CL}{(C0+CLn)}$$

$$\varepsilon' = \varepsilon\frac{C0+CLn}{(C0+C1+CLn)}$$

$$\left(1+\frac{\Delta F_L}{F_L}\right)^2 = \left(\frac{1}{1-\varepsilon}\right)\left(1-\varepsilon'\right)$$

$$= \left(1+\varepsilon+\varepsilon^2+\varepsilon^3+\ldots\right)\left(1-\varepsilon'\right)$$

$$= 1+\left(\varepsilon-\varepsilon'\right)+\varepsilon\left(\varepsilon-\varepsilon'\right)+\varepsilon^2\left(\varepsilon-\varepsilon'\right)+\varepsilon^3\left(\varepsilon-\varepsilon'\right)+\ldots$$

$$= 1+\left(\varepsilon-\varepsilon'\right)\left(1+\varepsilon+\varepsilon^2+\varepsilon^3+\ldots\right)$$

$$= 1+\varepsilon\left(1-\frac{(C0+CLn)}{(C0+C1+CLn)}\right)\left(1+\varepsilon+\varepsilon^2+\varepsilon^3+\ldots\right)$$

$$= 1+\left(\frac{C1}{C0+C1+CLn}\right)\varepsilon\left(1+\varepsilon+\varepsilon^2+\varepsilon^3+\ldots\right)$$

$$= 1+\left(\frac{C1}{C0+C1+CLn}\right)\frac{\varepsilon}{1-\varepsilon}$$

In the last expression, the well known closed form of an infinite geometric sum was substituted. Taking the square root shows that a simple solution for ΔFL/FL in terms of ΔCL/(C0+CLn) will be obtained if the usual MacLaurin expansion is used to pull the one from under the square root.

$$1+\frac{\Delta F_L}{F_L} = \sqrt{1+\left(\frac{C1}{C0+C1+CLn}\right)\frac{\varepsilon}{1-\varepsilon}}$$

We know that the square root approximation is going to be very good because a crystal mistuned by 1000 ppm relative to nominal is still in error by only 0.1%. Truncating a simple power expansion for the root at the first order will be entirely adequate for 0.1% accuracy. To begin we take advantage of the fact that C1 is small relative to (C0+C1+CL) regardless of the value of $\varepsilon = -\Delta CL/(C0+CLn)$. The expansion is shown below is to second order before truncation so that the error of a first order expansion can be evaluated. The exact tuning curve to first order is shown in Equation 6.

$$1+\frac{\Delta F_L}{F_L} = \sqrt{1+\left(\frac{C1}{C0+C1+CLn}\right)\frac{\varepsilon}{1-\varepsilon}}$$

$$= 1+\frac{1}{2}\left(\frac{C1}{C0+C1+CLn}\right)\frac{\varepsilon}{1-\varepsilon}-\frac{1}{4}\left(\frac{C1}{C0+C1+CLn}\right)^2\left(\frac{\varepsilon}{1-\varepsilon}\right)^2+\ldots$$

$$\frac{\Delta F_L}{F_L} \approx = -\frac{1}{2}\left(\frac{C1}{C0+C1+CLn}\right)\left(\frac{\dfrac{\Delta CL}{(C0+CLn)}}{1+\dfrac{\Delta CL}{(C0+CLn)}}\right) \qquad \textbf{Equation 6}$$

Error of the Series Truncation

The error of the approximation can be calculated to first order by taking the ratio of the second order correction to the first order correction as below.

$$Error = -\frac{\dfrac{1}{4}\left(\dfrac{C1}{C0+C1+CLn}\right)^2\left(\dfrac{\varepsilon}{1-\varepsilon}\right)^2}{\dfrac{1}{2}\left(\dfrac{C1}{C0+C1+CLn}\right)\left(\dfrac{\varepsilon}{1-\varepsilon}\right)} = \frac{1}{2}\left(\frac{C1}{C0+C1+CLn}\right)\left(\frac{\dfrac{\Delta CL}{(C0+CLn)}}{1+\dfrac{\Delta CL}{(C0+CLn)}}\right)$$

The magnitude of the first order error is the same as that of the first order approximation to the frequency error itself. For a 1000 ppm offset, the calculation is accurate to 1000ppm or an error in the calculated value of ΔF/FL of 1ppm. Similarly for a 100ppm frequency offset, the error is 10 ppb.

Comparison of the Exact Load Curve and the Tuning Sensitivity

As C1 is typically on the order of 10fF and CL is on the order of 10pF, there will be a very small error introduced in the exact load curve, Equation 6, by making the same approximation as was done for the tuning sensitivity; C1<<C0+CL. Making this approximation results in Equation 7and allows for the tuning sensitivity of Equation 4 and the load curve to be compared directly.

$$\frac{\Delta F_L}{F_L} = -\frac{1}{2}\left(\frac{C1}{C0+CLn}\right)\frac{\Delta CL}{(C0+CLn)}$$

$$\frac{\Delta F_L}{F_L} = -\frac{1}{2}\left(\frac{C1}{C0+CLn}\right)\frac{\dfrac{\Delta CL}{(C0+CLn)}}{\left(1+\dfrac{\Delta CL}{(C0+CLn)}\right)} \qquad \textbf{Equation 7}$$

When the approximation that 1 >> ΔCL/(C0+CLn) is introduced into Equation 7, the exact load curve reduces to the sensitivity load curve.

Note that load curves are tabulated by crystal vendors as frequency offset vs CL, not ΔCL. To plot these load curves for comparison to vendor data, plot the frequency offset vs CL = CLn + ΔCL.

The Tuning Curve

The exact tuning curve is derived as Equation 8 below from the load curve in the same manner as was done for the sensitivity based tuning curve. Both tuning curves are shown below for comparison.

$$\Delta CL = 2\frac{(C0+CLn)^2}{C1}\frac{\Delta F_L}{F_L}$$

$$\Delta CL = 2\frac{(C0+CLn)^2}{C1}\frac{\dfrac{\Delta F_L}{F_L}}{1+2\dfrac{(C0+CLn)}{C1}\dfrac{\Delta F_L}{F_L}} \qquad \textbf{Equation 8}$$
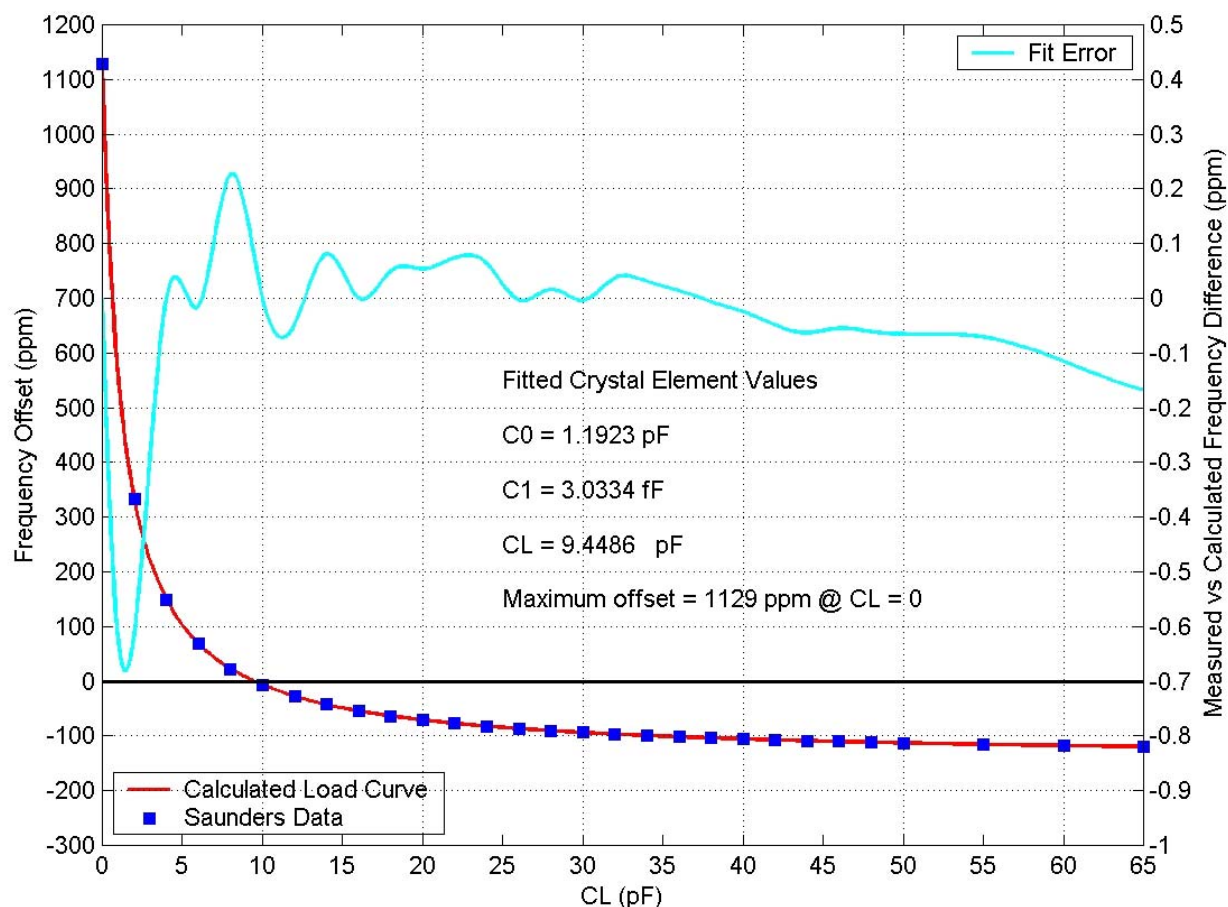
To use the tuning curve, the value of ΔCL that accounts of the fractional frequency error must be placed in parallel to the crystal. Since capacity can only be added to a circuit board, it is necessary that the circuit board layout result in less capacity than is necessary to tune the center frequency. Since oscillators used with parallel resonant crystals are inverting, the simplest implementation of the tuning capacitor is two caps, one each connected between a crystal terminal and ground. Due to the inversion of the oscillator, these two caps appear as a series combination across the crystal terminals. The advantage of this arrangement is that the series combination of two capacitors results in greater resolution of the parallel capacitance than from the standard values themselves. Typically once the parallel tuning capacity required is determined, this value is doubled and two caps of this value are connected to the crystal and ground–one cap per crystal terminal.

Experimental Verification of the Exact Load Curve

The load curve obtained appears plausible based on the model, but experimental verification is necessary for both the model and the derivation. Characterization data was taken with a Saunders and Associates 250B-1 test set for a sample 25MHz 3.2mm x 2.5 mm crystal. This test set provided ΔF/FL vs CL data at a 100µW drive level.
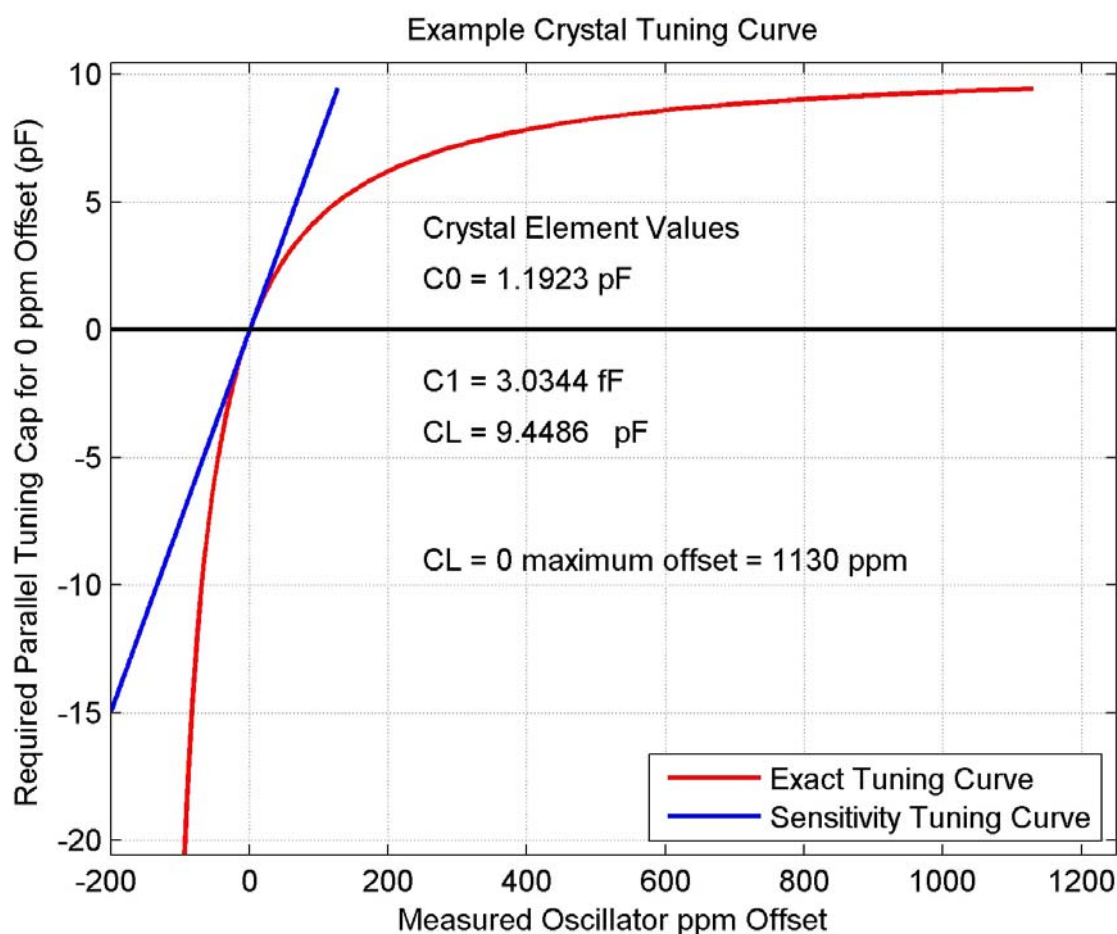
First the Saunders ΔF/FL vs CL data was plotted in Figure 1 along with the derived ΔF/FL vs CL curve obtained using an optimum set of crystal C0, C1 and CL values, fitted to Equation 6 using the Matlab *fminsearch* function. The results are plotted below in Figure 2 where it is seen that the fit between the exact load curve and the experimental data is very close. To make the correspondence easier to assess, the difference between the experimental data and Equation 8, which includes the C1 approximation, is also plotted. Based on these results the model provides a very compact description of the resonant frequency with excellent agreement with the data, particularly considering that ΔF/FL varies between 1129 ppm and -113 ppm while the load capacity varies from 0pF to 50pF. An added benefit is that this is a direct indication of the quality of the data taken by the Saunders test set.

Figure 2.   Saunders 250B-1 Experimental Data and Model Fit

Lastly the calculated Tuning Curve is plotted in Figure 3 with the Sensitivity Tuning Curve below for comparison.

Figure 3. Exact and Sensitivity Based Tuning Curves

Example Crystal Tuning Curve

Crystal Element Values
C0 = 1.1923 pF

C1 = 3.0344 fF
CL = 9.4486 pF

CL = 0 maximum offset = 1130 ppm

Conclusion

An expression, for all intents and purposes exact, has been derived for the tuning curve of a fundamental mode crystal. It is further shown that the exact curve is a very simple modification of the standard first order load curve based on differentiating the relationship between the series and parallel resonant mode frequencies derived from the standard model of crystal impedance.

The utility of the expression is most easily seen when designers tune crystal based clock generator for production. The exact expression coupled with C0, C1, CL and FL measured data allows for direct calculation of crystal load caps. More significant, but outside the scope of this article, is that since the load curve contains the FL for all load capacities, therefore it permits exact calculation of load caps to center VCXOs and predict the tuning range. Used properly, this new capability eliminates the errors designers often make when selecting VCXO crystals, such as not realizing the required Absolute Pull Range or APR.

Revision History

Rev.	Date	Originator	Description of Change
A	05/06/14	–	Initial release.
B	09/22/14	P. Wissell	1. Correct typo in the exact solution of Equation 8. 2. Add axes labels to Figure 3.

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